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Analytic Continuation of a Family of Dirichlet Series

by

Alexandru ZAHARESCU and Mohammad ZAKI

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Abstract. We study the meromorphic continuation of the Dirichlet series

$$F_{q,b,H,\alpha}(s) = \sum_{\substack{m,k\geq 1 \\ mk\equiv b \pmod q}} H(\alpha \log(m+k)) \frac{\Lambda(m)\Lambda(k)}{(m+k)^s}\,.$$

where Λ is the classical Von Mangoldt function, H is a smooth periodic function with period 1, $\alpha > 0$ is a real number, and $b, q > 0$ are integers with $(b, q) = 1$.

1. Introduction

The analytic continuation and the natural boundary of the series $\sum_p p^{-s}$, $s \in \mathbb{C}$, where p runs over the primes, were investigated by a number of authors including Kluyver [14], Landau [16], and Landau and Walfisz [17]. Considering a certain class of Dirichlet series which have Euler products, Estermann [10] gave a criterion for when the series can be continued to the whole plane, and when it has a natural boundary. The continuation and natural boundaries of Euler products were further studied in more general situations by Dahlquist [7] and Kurokawa[15]. A multi-variable generalization was recently discussed by Bhowmik, Essouabri and Lichtin [5]. The study of special values of the Riemann zeta function has motivated the study of multiple zeta values or multiple zeta functions defined by

$$\zeta(s_1, \ldots, s_r) = \sum_{1\leq n_1<n_2<\ldots<n_r} \frac{1}{n_1^{s_1}\cdots n_r^{s_r}}\,.$$

This series converges absolutely for $\Re(s_r) > 1$ and $\Re(s_i) \geq 1$ $(i = 1, 2, \ldots, r-1)$. The special values when $s_r \geq 2$ and $s_2, \ldots, s_{r-1} \geq 1$ with s_i integral for $1 \leq i \leq r$, have attracted special attention. The reader is referred to Cartier [6], Waldschmidt [23], and Zudilin [25] for treatments of the algebraic aspect of the theory, and Matsumoto [18] for an exposition of the analytic side. Zhao [24], and Akiyama, Egami, and Tanigawa [1] showed that $\zeta(s_1, \ldots, s_r)$ extends to a meromorphic function in $\mathbb{C}^r$, and explicitly described the

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location of its singularities. The multiple L-function defined by

$$L_k(s_1,\dots,s_k|\chi_1,\dots,\chi_k)=\sum_{1\le n_1<n_2<\dots<n_k}\frac{\chi(n_1)\cdots\chi(n_k)}{n_1^{s_1}\cdots n_k^{s_k}},$$

converges absolutely for $\Re(s_k)>1$ and $\Re(s_i)\ge 1$ $(i=1,2,\dots,k-1)$. Akiyama, Egami, and Tanigawa [2] showed that $L_k(s_1,\dots,s_r|\chi_1,\dots,\chi_k)$ extends to a meromorphic function in $\mathbb{C}^r$. Denote as usual $s=\sigma+it$, and let $P(x_1,\dots,x_r)$ be a polynomial with complex coefficients. Barnes [3], [4], and Mellin [20], [21] studied the multiple zeta-function

$$\zeta_r(s;P)=\sum_{m_1=1}^{\infty}\cdots\sum_{m_1=1}^{\infty}P(m_1,\dots,m_r)^{-s}.$$

Essouabri [9] showed that under certain assumptions the multi-variable generalization

$$\begin{aligned}&\zeta_r(s_1,\dots,s_n;P_1,\dots,P_n)\\ &=\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}P_1(m_1,\dots,m_r)^{-s_1}\cdots P_n(m_1,\dots,m_r)^{-s_n}\end{aligned}\tag{1.1}$$

has a meromorphic continuation to the whole space $\mathbb{C}^n$. The Euler-Zagier r-fold sum

$$\zeta_{EZ,r}(s_1,\dots,s_r)=\sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}m_1^{-s_1}(m_1+m_2)^{-s_2}\cdots(m_1+\dots+m_r)^{-s_r}\tag{1.2}$$

is a special case of (1.1), and has been studied in recent years. More generally, starting with the Dirichlet series

$$\psi_k(s)=\sum_{m=1}^{\infty}\frac{a_k(m)}{n^s},\ 1\le k\le r,$$

Matsumoto and Tanigawa [19] defined the multiple Dirichlet series

$$\Psi_r(s_1,\dots,s_r|\psi_1,\dots,\psi_r)=\sum_{m_1=1}^{\infty}\dots\sum_{m_r=1}^{\infty}\frac{a_1(m_1)}{m_1^{s_1}}\frac{a_2(m_2)}{(m_1+m_2)^{s_2}}\cdots\frac{a_r(m_r)}{(m_1+\dots+m_r)^{s_r}}$$

They showed that if $\psi_k(s)=\sum\limits_{m=1}^{\infty}\frac{a_k(m)}{n^s}$, $1\le k\le r$, are absolutely convergent for $\sigma=\Re(s)>\alpha_k>0$, can be continued meromorphically to the whole plane $\mathbb{C}$, holomorphic except for a possible pole of order at most 1 at $s=\alpha_k$, and of polynomial order in any fixed strip $\sigma_1\le\sigma\le\sigma_2$, then $\Psi_r(s_1,\dots,s_r|\psi_1,\dots,\psi_r)$ can be continued meromorphically to $\mathbb{C}^r$. Here, the location of the singularities is described explicitly. In particular, if all $\psi_k(s)$ are entire, then $\Psi_r(s_1,\dots,s_r|\psi_1,\dots,\psi_r)$ is also entire. The basic tool employed in the proof of this result is the Mellin-Barnes formula

$$(1+\lambda)^{-s}=\frac{1}{2\pi i}\int_{(c)}\frac{\Gamma(s-z)\Gamma(z)}{\Gamma(s)}\lambda^{-z}dz\tag{1.3}$$

where $s,\lambda\in\mathbb{C}$, $\lambda\ne 0$, $|arg\ \lambda|<\pi$, $\Re(s)>0$, $0<c<\Re(s)$, and the path of integration is the vertical line from $c-i\infty$ to $c+i\infty$. In [8], Egami and Matsumoto point out that if one assumes that $\psi_k(s)$ $1 \le k \le r$ have finitely many poles, then $\Psi_r(s_1, \ldots, s_r|\psi_1, \ldots, \psi_r)$ can be continued meromorphically to $\mathbb{C}^r$. But, if $\psi_k(s)$ $1 \le k \le r$ have infinitely many poles, then $\Psi_r(s_1, \ldots, s_r|\psi_1, \ldots, \psi_r)$ might have quite different behaviour.

Let $\Lambda(n)$ be the Von Mangoldt function, and let

$$M(s) = -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}\,.$$

Define

$$\phi_2(s) = \Psi(0, s; M, M) = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{\Lambda(k)\Lambda(m)}{(k+m)^s} = \sum_{n=1}^{\infty} \frac{G_2(n)}{n^s}\,,$$

where

$$G_2(n) = \sum_{k+m=n} \Lambda(k)\Lambda(m)\,.$$

Since $G_2(n) \le \sum_{k=1}^{n-1} \log(k)\log(n-k) \le n(\log n)^2$, $\phi_2(s)$ converges absolutely for $\Re(s) > 2$. In [8], some evidence was given that $\Re(s) = 1$ is the natural boundary for $\phi_2(s)$. Fujii [11] showed that under RH,

$$\sum_{n \le x} G_2(n) = \frac{1}{2}x^2 - H(x) + O((x\log x)^{\frac{4}{3}})\,,$$

where $H(x) = 2\sum_{\rho} \frac{x^{1+\rho}}{\rho(1+\rho)}$. In [8], Egami and Matsumoto proved the following result.

Assuming RH, $\phi_2(s)$ has a meromorphic continuation to $\Re(s) > 1$, and holomorphic except for the simple pole at $s = 2$ with residue 1, and $s = 1+\rho$ with residue $\frac{-2\eta(\rho)}{\rho}$ for every nontrivial zero ρ of $\zeta(s)$, where $\eta(\rho)$ is the multiplicity of ρ.

By Perron's formula,

$$\sum_{n \le x} G_2(n) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \phi_2(s)\frac{x^s}{s}ds + O(T^{-1}x^{2+\epsilon}),\ c > 2\,.$$

Shifting the path of integration to $\Re(s) = 1+\epsilon$, one finds that $\frac{1}{2}x^2 - H(x)$ equals the sum of the residues. Thus the properties of $H(x)$ are closely related to the behaviour of $\phi_2(s)$. It is expected (see Hardy and Littlewood [13]) that $G_2(n)$ for even n is approximately $nS_2(n)$, where

$$S_2(n) = \prod_{p|n}\left(1 + \frac{1}{p-1}\right)\prod_{(p,n)=1}\left(1 + \frac{1}{(p-1)^2}\right)\,.$$

Also, Montgomery and Vaughan [22] showed that

$$\sum_{n \le x} nS_2(n) = \frac{1}{2}x^2 + O(x\log x)\,.$$

Fujii's [11] formula can be written

$$\sum_{n\leq x}(G_2(n)-nS_2(n)) = -H(x)+O\left((x\log x)^{\frac{4}{3}}\right) .$$

Let $\mathcal{I}$ denote the set of imaginary parts of nontrivial zeros of $\zeta(s)$.

It is a well known conjecture that the elements of $\mathcal{I}$ are linearly independent over the rationals. The following is a special case of this conjecture.

(A) If $\gamma_j \in \mathcal{I}$ $(1 \leq j \leq 4)$, and $\gamma_1+\gamma_2 = \gamma_3+\gamma_4$ $(\neq 0)$, then (γ_3,γ_4) equals (γ_1,γ_2) or (γ_2,γ_1).

Fujii [12] studied additive properties of the zeros of $\zeta(s)$, and proved that the set

$$\{\gamma_1+\gamma_2 : \gamma_1,\gamma_2\in\mathcal{I}, \gamma_1>0, \gamma_2>0\}$$

is uniformly distributed modulo 1.

In [8], the following hypothesis is introduced.

(B) There exists a constant α, with $0<\alpha<\frac{\pi}{2}$, such that if $\gamma_j\in\mathcal{I}$ $(1\leq j\leq 4)$, $\gamma_1+\gamma_2\neq 0$, and (γ_3,γ_4) is neither equal to (γ_1,γ_2) nor to (γ_2,γ_1), then

$$|(\gamma_1+\gamma_2)-(\gamma_3+\gamma_4)| \geq \exp\left(-\alpha\left(|\gamma_1|+|\gamma_2|+|\gamma_3|+|\gamma_4|\right)\right) .$$

It is proved in [8] that under RH, and (B), $\Re(s)=1$ is the natural boundary of $\phi_2(s)$.

In this paper, for any positive integer q, any integer b relatively prime to q, any smooth periodic function H with period 1, and any real number $\alpha>0$, we consider the function

$$F_{q,b,H,\alpha}(s) = \sum_{\substack{m,k\geq 1\\ mk\equiv b \pmod q}} H(\alpha\log(m+k))\frac{\Lambda(m)\Lambda(k)}{(m+k)^s} .$$

We will prove the following result.

THEOREM 1. *Let b, and $q>0$ be integers with $(b,q)=1$. Let $H\in C^2(\mathbb{R})$ be periodic with period 1, and $\alpha>0$. Assuming GRH, $F_{q,b,H,\alpha}(s)$ has an analytic continuation to the half plane $\Re(s)>\frac{3}{2}$, except for simple poles at $s=2+2\pi i\alpha n$.*

It would be interesting to study the natural boundary of $F(s)$ for various choices of q, b, H, and α. Moreover, we prove under the same assumptions as in Theorem 1 that the difference of two such functions associated with two arithmetic progressions to the same modulus can be analytically continued to a larger region. To be precise, we have the following result.

THEOREM 2. *Let $H\in C^2(\mathbb{R})$ be periodic with period 1, and $\alpha>0$ be a real number. Let $(b_1,q)=1$, $(b_2,q)=1$. Assuming GRH, the difference $F_{q,b_1,H,\alpha}(s)-F_{q,b_2,H,\alpha}(s)$ has an analytic continuation to the half-plane $\Re(s)>1$.*

2. Proofs of Theorems 1 and 2

In this setion we prove Theorem 1 and Theorem 2. To proceed we start with some terminology.

For any positive integer q, any smooth periodic function H, and any real number $\alpha > 0$, let

$$F_{G,H,\alpha}(s) = \sum_{m,k\geq 1} G(mk)H(\alpha \log(m+k))\frac{\Lambda(m)\Lambda(k)}{(m+k)^s}\,,$$

where $G : \mathbb{Z} \to \mathbb{C}$ is either the constant function 1, or a Dirichlet character modulo the prime q. Assume that H has period 1, $H \in C^2(\mathbb{R})$, and

$$H(t) = \sum_{n\in\mathbb{Z}} c_n e(nt),\ \ c_n = \int_0^1 H(t)e(-nt)dt\,.$$

Here, we have that

$$c_n \ll_H \frac{1}{n^2}\,.$$

For a Dirichlet character χ modulo the prime q, let

$$\phi_2(s,\chi) = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{\Lambda(k)\Lambda(m)\chi(k)\chi(m)}{(k+m)^s} = \sum_{n=1}^{\infty}\frac{G_2(n,\chi)}{n^s}\,,$$

where

$$G_2(n,\chi) = \sum_{k+m=n}\Lambda(k)\Lambda(m)\chi(k)\chi(m)\,.$$

Observe that by interchanging the order of summation, we obtain

$$\begin{aligned} F_{\chi,H,\alpha}(s) &= \sum_{n\in\mathbb{Z}} c_n \sum_{m,k\geq 1}\chi(m)\chi(k)\frac{\Lambda(m)\Lambda(k)}{(m+k)^s}e^{2\pi i n\alpha\log(m+k)} \\ &= \sum_{n\in\mathbb{Z}} c_n \sum_{m,k\geq 1}\chi(m)\chi(k)\frac{\Lambda(m)\Lambda(k)}{(m+k)^{s-2\pi i\alpha n}}\,. \end{aligned}$$

Thus, we have that

$$F_{\chi,H,\alpha}(s) = \sum_{n\in\mathbb{Z}} c_n\phi_2(s-2\pi i\alpha n,\chi)\,.$$

Next, we prove the following result.

THEOREM 3. *Let q be a positive integer, and let χ be a primitive character modulo q. Under GRH,*

$$\phi_2(s,\chi) = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{\Lambda(k)\Lambda(m)\chi(k)\chi(m)}{(k+m)^s} = \sum_{n=1}^{\infty}\frac{G_2(n,\chi)}{n^s}\,,$$

where

$$G_2(n,\chi) = \sum_{k+m=n}\Lambda(k)\Lambda(m)\chi(k)\chi(m)\,,$$

can be continued holomorphically to the half-plane $\Re s > 1$.

Proof. Let I_χ denote the set of all imaginary parts of nontrivial zeros of $L(s,\chi)$. Since $|G_2(n,\chi)| \le \sum_{k=1}^{n-1} \log(k)\log(n-k) \le n(\log n)^2$, we have that $\phi_2(s,\chi)$ converges absolutely for $\Re(s) > 2$.

Let $\delta > 0$, $s = \sigma + it$, and assume that $\Re s > 2 + 2\delta$. We may write $\phi_2(s,\chi)$ as

$$\phi_2(s,\chi) = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{\Lambda(k)\chi(k)\Lambda(m)\chi(m)}{k^s}\left(1+\frac{m}{k}\right)^{-s}.$$

Applying Mellin-Barnes formula (1.3) with $\lambda = \frac{m}{k}$, we obtain

$$\phi_2(s,\chi) = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{\Lambda(k)\chi(k)\Lambda(m)\chi(m)}{k^s}\frac{1}{2\pi i}\int\limits_{(c=1+\delta)} \frac{\Gamma(s-z)\Gamma(z)}{\Gamma(s)}\left(\frac{m}{k}\right)^{-z} dz.$$

By changing the order of summation and integration, we find that

$$\phi_2(s,\chi) = \frac{1}{2\pi i}\int\limits_{(c=1+\delta)} \frac{\Gamma(s-z)\Gamma(z)}{\Gamma(s)}\sum_{k=1}^{\infty}\Lambda(k)\chi(k)k^{z-s}\sum_{m=1}^{\infty}\Lambda(m)\chi(m)m^{-z}dz. \tag{2.1}$$

The condition $0 < c < \sigma$ is necessary to apply (1.3). This together with the fact that the two series inside the integrand in (2.1) are absolutely and uniformly convergent when $\sigma - c > 1$ and $c > 1$ justifies the choice of c. We further write

$$\phi_2(s,\chi) = \frac{1}{2\pi i}\int\limits_{(c=1+\delta)} \frac{\Gamma(s-z)\Gamma(z)}{\Gamma(s)}M(s-z,\chi)M(z,\chi)dz,$$

where

$$M(S,\chi) = \sum_{m=1}^{\infty}\Lambda(m)\chi(m)m^{-S} = -\frac{L'(S,\chi)}{L(S,\chi)}. \tag{2.2}$$

Next, we shift the path of integration from $\Re z = 1+\delta$ to $\Re z = -\delta$. Recall that $N(T,\chi)$, $T > 0$, denotes the number of zeros of $L(s,\chi)$ in the region $0 < \sigma < 1$, $-T \le t \le T$. Also,

$$N(T+1,\chi) - N(T,\chi) \ll \log T \text{ for } T \ge 2.$$

So, we may find T with arbitrarily large absolute value $|T|$ such that

$$|T-\gamma| \gg (\log T)^{-1} \text{ for any } \gamma \in I_\chi. \tag{2.3}$$

Also recall that the formula

$$M(z,\chi) = -\sum_{\substack{|y-\gamma|<1\\ \gamma\in I_\chi\\ \rho=\frac{1}{2}+i\gamma}} \frac{1}{z-\rho} + O(\log(|y|+2)), \tag{2.4}$$

where $y = \Im z$, and $\gamma = \Im\rho$, holds uniformly for $-1 \le x = \Re z \le 2$. By combining (2.3) and (2.4), we see that if T satisfies (2.3), then for $z = x+iT$, $-1 \le x \le 2$, we get

$$M(z,\chi) \ll (\log T)^2.$$

Also,

$$M(s-z,\chi) = O_\delta(1) \text{ for } -\delta \le x \le 1+\delta\,,$$

since $\Re s > 2+2\delta,\ \delta > 0$. Note that Stirling's formula applied to the following integral with T satisfying (2.3) yields

$$\int_{-\delta+iT}^{1+\delta+iT} \frac{\Gamma(s-z)\Gamma(z)}{\Gamma(s)} M(s-z,\chi)M(z,\chi)dz$$

$$\ll \frac{(|t-T|+1)^{\sigma-\frac{1}{2}}}{(|t|+1)^{\sigma-\frac{1}{2}}T^{\frac{1}{2}}}(\log T)^2 \exp\left(-\frac{\pi}{2}(|T|+|t-T|-|t|)\right)\int_{-\delta}^{1+\delta}(|t-T|+1)^{-x}T^x dx\,,$$

where the right side tends to 0 as T tends to ∞. Similarly, the integral

$$\int_{-\delta-iT}^{1+\delta-iT} \frac{\Gamma(s-z)\Gamma(z)}{\Gamma(s)} M(s-z,\chi)M(z,\chi)dz$$

tends to 0 as T tends to ∞. So we have that the above shifting of the path of integration is justified. We encounter, in the process, the poles $z=\rho$, for each nontrivial zero ρ of $L(s,\chi)$, and $z=0$. The residue at a pole $z=\rho$ is given by

$$-n(\rho)\frac{\Gamma(s-\rho)\Gamma(\rho)}{\Gamma(s)}M(s-\rho,\chi)\,,$$

where $n(\rho)$ is the multiplicity. At $z=0$, the residue is given by

$$\begin{cases} M(s,\chi)M(0,\chi) & \text{if } \chi(-1)=-1\,,\\ -M(s,\chi)\,[a_0(\Gamma)-b(\chi)] & \text{if } \chi(-1)=1\,,\end{cases}$$

where

$$\begin{aligned} a_0(\Gamma) &= \lim_{z\to 0}\left[\Gamma(z)-\frac{1}{z}\right]\\ &= \sum_{n=0}^{\infty}\frac{(-)^n}{n!n} + \int_1^{\infty}\exp(-t)t^{-1}dt\,,\end{aligned}$$

and for $\chi(-1)=1$,

$$\frac{L'(s,\chi)}{L(s,\chi)} = \frac{1}{s} + b(\chi) + \ldots.$$

Thus if $\chi(-1) = -1$, then

$$
\begin{aligned}
\phi_2(s,\chi) = &- \sum_{\substack{\rho=\frac{1}{2}+i\gamma \\ \gamma\in I_\chi}} \frac{\Gamma(s-\rho)\Gamma(\rho)}{\Gamma(s)} M(s-\rho,\chi) \\
&+ \frac{1}{2\pi i}\int_{(-\delta)} \frac{\Gamma(s-z)\Gamma(z)}{\Gamma(s)} M(s-z,\chi)M(z,\chi)dz \\
&+ M(s,\chi)M(0,\chi)\,.
\end{aligned}
\tag{2.5}
$$

Similarly, if $\chi(-1) = 1$, then

$$
\begin{aligned}
\phi_2(s,\chi) = &- \sum_{\substack{\rho=\frac{1}{2}+i\gamma \\ \gamma\in I_\chi}} \frac{\Gamma(s-\rho)\Gamma(\rho)}{\Gamma(s)} M(s-\rho,\chi) \\
&+ \frac{1}{2\pi i}\int_{(-\delta)} \frac{\Gamma(s-z)\Gamma(z)}{\Gamma(s)} M(s-z,\chi)M(z,\chi)dz \\
&- M(s,\chi)\,[a_0(\Gamma) - b(\chi)]\,.
\end{aligned}
\tag{2.6}
$$

In order to continue $\phi_2(s,\chi)$ holomorphically to $\Re s > 1$, we make use of the last two equations. The last terms in both (2.5) and (2.6) are clearly meromorphic on the whole complex plane. The poles of these terms coincide with the poles of $M(s,\chi)$. The integral

$$
\frac{1}{2\pi i}\int_{(-\delta)} \frac{\Gamma(s-z)\Gamma(z)}{\Gamma(s)} M(s-z,\chi)M(z,\chi)dz
$$

is convergent uniformly in any compact subset of the half-plane $\Re s > 1-\delta$, and hence it is holomorphic in that half-plane.

Next, consider the first term,

$$
B_2(s) = -\sum_{\rho} \frac{\Gamma(s-\rho)\Gamma(\rho)}{\Gamma(s)} M(s-\rho,\chi)\,.
$$

$\Gamma(s-\rho)$ has poles at $s = \rho - l$ $(l = 0, 1, 2, \ldots)$. If $\chi(-1) = 1$, then $M(s-\rho,\chi)$ has poles at $s = \rho$ and $s = \rho + \rho'$, where ρ, and ρ' denote nontrivial zeros of $L(s,\chi)$. If $\chi(-1) = -1$, then $M(s-\rho,\chi)$ has poles at $s = \rho + \rho'$, where ρ and ρ' denote nontrivial zeros of $L(s,\chi)$. Assuming GRH, we see that there are no poles of $B_2(s)$ in the region $\Re s > 1$. We conclude that $\phi_2(s,\chi)$ is holomorphic in $\Re s > 1$.

LEMMA 1. *Let q be a positive integer, and let χ_0 be the principal character modulo q. Under GRH,*

$$
\phi_2(s,\chi_0) = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{\Lambda(k)\Lambda(m)\chi_0(k)\chi_0(m)}{(k+m)^s}
$$

can be continued meromorphically to the half-plane $\Re s > 1$. It is holomorphic except for the simple poles at $s = 2$ with residue 1 and $s = 1+\rho$ with residue $\frac{2n(\rho)}{\rho}$ for any nontrivial zero ρ of $\zeta(s)$, where $n(\rho)$ is the multiplicity of ρ.

Proof. Let $g(s) = \phi_2(s) - \phi_2(s, \chi_0)$. Then

$$g(s) = \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \frac{\Lambda(m_1)\Lambda(m_2)(1-\chi_0(m_1 m_2))}{(m_1+m_2)^s} = \sum_{\substack{m_1,m_2\geq 1\\(q,m_1m_2)>1}} \frac{\Lambda(m_1)\Lambda(m_2)}{(m_1+m_2)^s}.$$

We write

$$g(s) = \Sigma_1 + \Sigma_2 + \Sigma_3,$$

where

$$\Sigma_1 = \sum_{\substack{m_1,m_2\geq 1\\(q,m_1)>1\\(q,m_2)=1}} \frac{\Lambda(m_1)\Lambda(m_2)}{(m_1+m_2)^s},$$

$$\Sigma_2 = \sum_{\substack{m_1,m_2\geq 1\\(q,m_1)=1\\(q,m_2)>1}} \frac{\Lambda(m_1)\Lambda(m_2)}{(m_1+m_2)^s},$$

and

$$\Sigma_3 = \sum_{\substack{m_1,m_2\geq 1\\(q,m_1)>1\\(q,m_2)>1}} \frac{\Lambda(m_1)\Lambda(m_2)}{(m_1+m_2)^s}.$$

Assume that q has prime factorization

$$q = p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}.$$

Next, consider Σ_1. Then,

$$\begin{aligned}\Sigma_1 &= \sum_{1\leq j\leq t} \sum_{l\geq 1} \sum_{\substack{m_2\geq 1\\(q,m_2)=1}} \frac{\Lambda(p_j^l)\Lambda(m_2)}{(p_j^l+m_2)^s}\\ &= \sum_{1\leq j\leq t} \log p_j \sum_{l\geq 1} \sum_{\substack{m_2\geq 1\\(q,m_2)=1}} \frac{\Lambda(m_2)}{(p_j^l+m_2)^s}\\ &= \sum_{1\leq j\leq t} \log p_j \sum_{n\geq 2} \frac{b(n)}{n^s},\end{aligned}$$

where

$$b(n) = \sum_{\substack{n=p_j^l+m\\ l\geq 1,(m,q)=1}} \Lambda(m) \leq \sum_{1\leq l\leq \frac{\log n}{\log p_j}} \log n \ll_q (\log n)^2.$$

Consequently, $\sum_{n\geq 2} \frac{b(n)}{n^s}$ converges for $\Re s > 1$.

Similarly, Σ_2 and Σ_3 converge for $\Re s > 1$, proving the lemma.

LEMMA 2. *Let q be a positive integer, and let χ be a nonprincipal character modulo q. Under RH,*

$$\phi_2(s,\chi)=\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{\Lambda(k)\Lambda(m)\chi(k)\chi(m)}{(k+m)^s}$$

can be continued holomorphically to the half-plane $\Re s>1$.

Proof. We know that the lemma holds for primitive charaters. Let χ be a nonprincipal character with modulus q. Let $d|q$ be the conductor of χ. Then, $\chi(n)=\tau(n)\chi_0(n)$ for all n, where χ_0 is the principal character with modulus q, and τ is a primitive character modulo the conductor d of χ. Now, $\tau(n)-\chi(n)=\tau(n)(1-\chi_0(n))$. Let $g(s)=\phi(s,\tau)-\phi(s,\chi)$. Then,

$$\begin{aligned}g(s)&=\sum_{m_1,m_2\geq 1}\frac{\Lambda(m_1)\Lambda(m_2)}{(m_1+m_2)^s}\tau(m_1m_2)(1-\chi_0(m_1m_2))\\&=\sum_{\substack{m_1,m_2\geq 1\\(q,m_1m_2)>1}}\frac{\Lambda(m_1)\Lambda(m_2)\tau(m_1m_2)}{(m_1+m_2)^s}\,.\end{aligned}$$

We write

$$g(s)=\Sigma_1+\Sigma_2+\Sigma_3\,,$$

where

$$\Sigma_1=\sum_{\substack{m_1,m_2\geq 1\\(q,m_1)>1\\(q,m_2)=1}}\frac{\Lambda(m_1)\Lambda(m_2)\tau(m_1m_2)}{(m_1+m_2)^s}\,,$$

$$\Sigma_2=\sum_{\substack{m_1,m_2\geq 1\\(q,m_2)>1\\(q,m_1)=1}}\frac{\Lambda(m_1)\Lambda(m_2)\tau(m_1m_2)}{(m_1+m_2)^s}\,,$$

and

$$\Sigma_3=\sum_{\substack{m_1,m_2\geq 1\\(q,m_1)>1\\(q,m_2)>1}}\frac{\Lambda(m_1)\Lambda(m_2)\tau(m_1m_2)}{(m_1+m_2)^s}\,.$$

By a similar argument to the one employed in the previous lemma, each of the terms Σ_1, Σ_2, Σ_3 converges for $\Re s>1$. This proves the lemma.

Proof of Theorem 1. Let q, b, α, H be as in the statement of Theorem 1, and consider

$$F_{q,b,H,\alpha}(s)=\sum_{\substack{m,k\geq 1\\mk\equiv b\pmod q}}\frac{\Lambda(m)\Lambda(k)}{(m+k)^s}H(\alpha\log(m+k))\,.$$

We want to show that assuming GRH, $F_{q,b,H,\alpha}(s)$ is analytic on the half plane $\Re(s) > \frac{3}{2}$, except for simple poles at $s = 2 + 2\pi i\alpha n$. We have that

$$\begin{aligned} F_{\chi,H,\alpha}(s) &= \sum_{n\in\mathbb{Z}} c_n \sum_{m_1,m_2\geq 1} \chi(m_1m_2)\frac{\Lambda(m_1)\Lambda(m_2)}{(m_1+m_2)^s} e^{2\pi i n\alpha \log(m_1+m_2)} \\ &= \sum_{n\in\mathbb{Z}} c_n \sum_{m_1,m_2\geq 1} \chi(m_1m_2)\frac{\Lambda(m_1)\Lambda(m_2)}{(m_1+m_2)^{s-2\pi i\alpha n}} \\ &= \sum_{n\in\mathbb{Z}} c_n \phi_2(s-2\pi i\alpha n,\chi) \end{aligned}$$

We first show that $\sum\limits_{n\in\mathbb{Z}} c_n\phi_2(s-2\pi i\alpha n,\chi)$ converges absolutely and uniformly in $\sigma \geq \sigma_1 > 2$ for any $\sigma_1 > 2$. Note that

$$\phi_2(s,\chi) = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \frac{\Lambda(k)\Lambda(m)\chi(k)\chi(m)}{(k+m)^s} = \sum_{n=1}^{\infty} \frac{G_2(n,\chi)}{n^s},$$

where

$$G_2(n,\chi) = \sum_{k+m=n} \Lambda(k)\Lambda(m)\chi(k)\chi(m).$$

Since $|G_2(n,\chi)| \leq \sum\limits_{k=1}^{n-1} \log(k)\log(n-k) \leq n(\log n)^2$, $|\phi_2(s,\chi)| \leq \sum\limits_{m=2}^{\infty} \frac{m\log^2 m}{m^\sigma}$. So, if $\sigma \geq \sigma_1 > 2$, we see that for all s with $\sigma \geq \sigma_1$,

$$|\phi_2(s)| \leq C_{\sigma_1}$$

for some constant C_{σ_1} depending only on σ_1.

Since $\sum\limits_{n\in\mathbb{Z}} |c_n| < \infty$, the sum

$$F_{\chi,H,\alpha}(s) = \sum_{n\in\mathbb{Z}} c_n\phi_{2,\chi}(s-2\pi i\alpha n)$$

converges absolutely and uniformly in $\sigma \geq \sigma_1 > 2$. Hence, $F_{\chi,H,\alpha}(s)$ is holomorphic in $\Re s = \sigma > 2$. Let us now recall the following lemma from [8].

LEMMA 3. *Let $\epsilon > 0$. Assuming RH, the estimate*

$$\phi_2(s) \ll T^{\frac{1}{2}}(\log T)^2 \tag{2.7}$$

holds for $s = \sigma + iT$, $1+\epsilon \leq \sigma \leq \frac{3}{2}-\epsilon$, or $\frac{3}{2}+\epsilon \leq \sigma \leq 2+\epsilon$.

Since $\phi_2(s) - \phi_2(s,\chi_0)$ is holomorphic in $\Re s > 1$, the above estimate also holds for $\phi_2(s,\chi_0)$. We also have the following estimate whose proof is very similar to the proof of Lemma 3.

LEMMA 4. *Let $\epsilon > 0$, and let χ be a nonprincipal character modulo a positive integer q. Assuming GRH, the estimate*

$$\phi_2(s,\chi) \ll_{q,\epsilon} T^{\frac{1}{2}}(\log T)^2 \tag{2.8}$$

holds for $s = \sigma + iT$, $1+\epsilon \leq \sigma \leq 2+\epsilon$.

Thus, for any nonprincipal character χ and $s = \sigma + iT$ with $1 + \epsilon \leq \sigma \leq 2 + \epsilon$, we get that

$$\phi_2(s - 2\pi i\alpha n, \chi) \ll_\epsilon (T + 2\pi\alpha|n|)^{\frac{1}{2}}(\log(T + 2\pi\alpha|n|))^2 \ll_{q,\epsilon,\eta} (T + 2\pi\alpha|n|)^{\frac{1}{2}+\eta}$$

for any $\eta > 0$. Fix $T > 0$. Let $s = \sigma + iT'$, $|T'| \leq T$, $1 + \epsilon \leq \sigma \leq 2 + \epsilon$. Write

$$F_{\chi,H,\alpha}(s) = c_0\phi_2(s, \chi) + \sum_{n=1}^{\infty} f_n(s)\,,$$

where $f_n(s) = c_n\phi_2(s - 2\pi i\alpha n, \chi) + c_{-n}\phi_2(s + 2\pi i\alpha n, \chi)$. Now,

$$f_n(s) \ll_{q,\epsilon,\eta} (|c_n| + |c_{-n}|)(T' + 2\pi\alpha|n|)^{\eta+\frac{1}{2}}\,.$$

Since $|c_k| \ll \frac{1}{k^2}$ for any nonzero integer k, we have that $\sum\limits_{n=1}^{\infty} f_n(s)$ converges uniformly for all $s = \sigma + iT'$, $|T'| < T$, $1 + \epsilon \leq \sigma \leq 2 + \epsilon$.

We conclude that if χ is nonprincipal, then $F_{\chi,H,\alpha}(s)$ is holomorphic in $\Re s > 1$, Similarly, it follows that

$$F_{1,H,\alpha}(s) = \sum_{m_1\geq 1}\sum_{m_2\geq 1} H(\alpha\log(m_1 + m_2))\frac{\Lambda(m_1)\Lambda(m_2)}{(m_1 + m_2)^s}$$

is holomorphic in $\Re s > \frac{3}{2}$, except for simple poles at $s = 2 + 2\pi i\alpha n$ for each $n \in \mathbb{Z}$, and

$$F_{\chi_0,H,\alpha}(s) = \sum_{m_1\geq 1}\sum_{m_2\geq 1} H(\alpha\log(m_1 + m_2))\frac{\Lambda(m_1)\Lambda(m_2)\chi_0((m_1m_2)}{(m_1 + m_2)^s}$$

is holomorphic in $\Re s > \frac{3}{2}$, except for simple poles at $s = 2 + 2\pi i\alpha n$ for each $n \in \mathbb{Z}$.

Lastly, we have that

$$\begin{aligned}F_{q,b,H,\alpha}(s) &= \sum_{\substack{m,k\geq 1\\ mk\equiv b \pmod q}} \frac{\Lambda(m)\Lambda(k)}{(m + k)^s} H(\alpha\log(m + k))\\ &= \frac{1}{\phi(q)}\sum_{\chi \pmod q} \overline{\chi}(b)F_{\chi,H,\alpha}(s)\,.\end{aligned}$$

Since the right side has the desired analytic properties, we conclude that the theorem holds.

Proof of Theorem 2. Let $H \in C^2(\mathbb{R})$ be periodic with period 1, and $\alpha > 0$. Let $(b_1, q) = 1$, $(b_1, q) = 1$. Assume GRH. Note that

$$\begin{aligned}F_{q,b_1,H,\alpha}(s) - F_{q,b_2,H,\alpha}(s) &= \frac{1}{\phi(q)}\sum_{\chi \pmod q} \overline{\chi}(b_1)F_{\chi,H,\alpha}(s)\\ &\quad - \frac{1}{\phi(q)}\sum_{\chi \pmod q} \overline{\chi}(b_2)F_{\chi,H,\alpha}(s)\\ &= \frac{1}{\phi(q)}\sum_{\substack{\chi \pmod q\\ \chi\neq\chi_0}} (\overline{\chi}(b_1) - \overline{\chi}(b_2))\, F_{\chi,H,\alpha}(s)\,.\end{aligned}$$

Since here the terms involving the principal character cancel, it follows from the proof of Theorem 1 that the last sum above is analytic in the half plane $\Re(s) > 1$. This completes the proof of Theorem2.

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Alexandru Zaharescu, Institute of Mathematics of the Romanian Academy, P. O. Box 1–764, Bucharest 014700, Romania, and Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 W. Green Street,
Urbana, IL, 61801, USA
e-mail: zaharesc@math.uiuc.edu

Mohammad Zaki, Department of Mathematics, University of Illinois at Urbana Champaing, 1409 W. Green Street, Urbana, IL, 61801, USA
e-mail: mzaki@math.uiuc.edu